(44) 5.3 THE FUNDAMENTAL THEOREM OF CALCULUS (D)

Example 1 30 Jan. 12. 2008	Show that $f(x)$ is an increasing function on $[1, \infty)$, where $f(x) = \int_1^{\sqrt{x}} \sqrt{1+t^4}\, dt + \int_0^{\pi} x^3 \sin^2(x^2+1)\, dx$

Solution

$$f(x) = \int_1^{\sqrt{x}} \sqrt{1+t^4}\, dt + \int_0^{\pi} x^3 \sin^2(x^2+1)\, dx$$

$$f'(x) = D_x \int_1^{\sqrt{x}} \sqrt{1+t^4}\, dt + D_x \int_0^{\pi} x^3 \sin^2(x^2+1)\, dx = \sqrt{1+x^2} \cdot \frac{1}{2\sqrt{x}} + 0 = \frac{\sqrt{1+x^2}}{2\sqrt{x}} > 0 \text{ for } x \geq 1$$

$f \nearrow$ on $[1, \infty)$

Example 2 29 June 4. 2007	Let $F(x) = \int_1^{x^3} \cos^3 u\, du + \int_1^{x^3+x} \sqrt{a+s^4}\, ds + \int_{x^3}^{4} \cos^3 u\, du$ Show that F is an increasing function

Solution

$$F(x) = \int_1^{x^3} \cos^3 u\, du + \int_1^{x^3+x} \sqrt{a+s^4}\, ds + \int_{x^3}^{4} \cos^3 u\, du$$

$$F'(x) = \cos^3 x^3 \cdot 3x^2 + \sqrt{a+(x^3+x)^4} \cdot (3x^2+1) + 0 - \cos x^3 \cdot (3x^2) = \sqrt{a+(x^3+x)^4} \cdot (3x^2+1) > 0$$

$\therefore$ F is increasing

Example 3 34 June 21, 2009	Let $f(x) = \int_0^{x} (t^2-t+11)^{20}\, dt + \int_0^{1} (x^2-x+11)^{20}\, dx.$ Show that $f(x)$ is increasing

Solution

$$f(x) = \int_0^{x} (t^2-t+11)^{20}\, dt + \int_0^{1} (x^2-x+11)^{20}\, dx.$$

$$f'(x) = D_x \int_0^{x} (t^2-t+11)^{20}\, dt + D_x \int_0^{1} (x^2-x+11)^{20}\, dx.$$

$$= (x^2-x+11)^{20}\,(1) + 0 = (x^2-x+11)^{20}\,(1) > 0$$

$\therefore$ f is increasing

Example 4 28 January 13. 2007	Find the equation of the tangent line to the curve $y = 7 + \int_{3}^{4x-x^2} \frac{t}{t^2+1} dt$ at $x = 1$

Solution

$$y = 7 + \int_{3}^{4x-x^2} \frac{t}{t^2+1} dt$$

$$y^{\backslash} = D_x \int_{3}^{4x-x^2} \frac{t}{t^2+1} dt = \frac{4x - x^2}{(4x-x^2)^2 + 1} \cdot (4 - 2x)$$

$$m = y^{\backslash}\big|_{x=1} = \frac{4-1}{(4-1)^2+1} \cdot (4-2) = \frac{3}{10} \cdot 2 = \frac{3}{5}$$

$$y\big|_{x=1} = 7 + \int_{3}^{3} \frac{t}{t^2+1} dt = 7 + 0 = 7$$

$$m = \frac{3}{5} \qquad , \qquad p(1,7)$$

$$y - y_1 = m(x - x_1)$$

$$y - 7 = \frac{3}{5}(x - 1)$$

$$5y - 35 = 3x - 3$$

$$3x - 5y + 32 = 0$$

Example 5 31 June 5, 2008	Find the local Extrema of $f(x) = \int_{0}^{x^2} \frac{1}{\sin t + 5} dt \quad x \in \mathcal{R}$

Solution

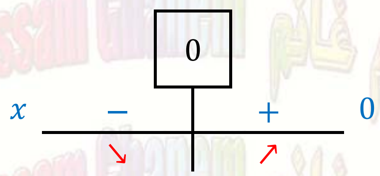
$$f(x) = \int_{0}^{x^2} \frac{1}{\sin t + 5} dt$$

$$f^{\backslash}(x) = \frac{1}{\sin x^2 + 5} \cdot 2x = \frac{2x}{\sin x^2 + 5}$$

$$f^{\backslash}(x) = 0$$

$$2x = 0 \quad \longrightarrow \quad x = 0$$

$$f(0) = \int_{0}^{0} \frac{1}{\sin t + 5} dt = 0$$



$\therefore$ mini. local extrema at $(1,0)$

Example 6

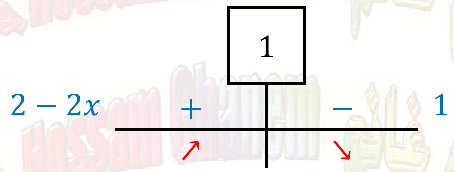
36 January 17, 2010

Let $f(x) = \int_{1}^{2x-x^2} \frac{1}{t^4+2}\, dt$ Find the local Maximum of $f(x)$.

Solution

$$f(x) = \int_{1}^{2x-x^2} \frac{1}{t^4+2}\, dt$$

$$f'(x) = \frac{1}{(2x-x^2)^4+2} \cdot (2-2x) = \frac{2-2x}{(2x-x^2)^4+2}$$



$$f'(x) = 0$$

$$2 - 2x = 0 \qquad \rightarrow \qquad x = 1$$

$$f(1) = \int_{1}^{1} \frac{1}{t^4+2}\, dt = 0$$

$\therefore$ 0 is maxi. local extrema at (1,0)

Example 7

33 January 20, 2009

Find the $x-$ coordinates of the Points of inflection of the continuous function:

$$f(x) = \int_{-5}^{x^2} \frac{1}{3+t^2}\, dt + \int_{1}^{7} \sqrt{3+t^2}\, dt.$$

Solution

$$f(x) = \int_{-5}^{x^2} \frac{1}{3+t^2}\, dt + \int_{1}^{7} \sqrt{3+t^2}\, dt.$$

$$f'(x) = D_x \int_{-5}^{x^2} \frac{1}{3+t^2}\, dt + D_x \int_{1}^{7} \sqrt{3+t^2}\, dt. = \frac{2x}{3+x^4} + 0 = \frac{2x}{3+x^4}$$

$$f''(x) = \frac{(3+x^4)\cdot 2 - 2x(4x^3)}{(3+x^4)^2} = \frac{2x^4 - 8x^4 + 6}{(3+x^4)^2} = \frac{6-6x^4}{(3+x^4)^2}$$

$$f''(x) = 0$$

$$6 - 6x^4 = 6(1-x^4) = 6(1-x^2)(1+x^2)$$

$$= 6(1-x)(1+x)(1+x^2)$$

$\therefore$ inflection point at $x = -1$, $x = 1$

Example 8 37 June 6, 2010	Find the point(s) of inflection, if any, of the curve $y = \int_0^x \frac{t^2 - 1}{t^2 + 1} dt.$

Solution

$$y = \int_0^x \frac{t^2 - 1}{t^2 + 1} dt$$

$$y^{\backslash} = D_x \int_0^x \frac{t^2 - 1}{t^2 + 1} dt = \frac{x^2 - 1}{x^2 + 1} (1)$$

$$y^{\backslash\backslash} = \frac{(x^2 + 1) \cdot 2x - (x^2 - 1) \cdot 2x}{(x^2 + 1)^2} = \frac{2x[x^2 + 1 - x^2 + 1]}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$$

$$y^{\backslash\backslash} = 0$$

$$4x = 0 \qquad \rightarrow \qquad x = 0$$

$$y|_{x=0} = \int_0^0 \frac{t^2 - 1}{t^2 + 1} dt = 0$$

Inflection point at $(0, 0)$

Homework

1	Given $f(x)=\int_0^x (t^2+1)dt$ for $x>0$ Show that $f(x)$ is increasing function for all $x>0$	10 June 6, 1994
2	Show that the function $f(x)=\int_o^x \sqrt{t^{\frac{2}{3}}+\cos^4 t}\ dt$ is increasing on $\left[0\ ,\frac{\pi}{2}\right]$	9 January 8, 1994
3	Show that the function $f(x)=\int_{\frac{1}{x}}^{\sqrt{x}} \sqrt{t^4+1}\,dt$ is increasing on $(0\ ,\infty)$	7 June 17, 1993
4	Let $f(x)=\int_1^{x^3+x} \sqrt{7+t^4}\,dt+\int_3^5 \cos^7 w\,dw$ show that f is an increasing function on $\mathcal{R}$	21 May 27. 2001
5	Show that $f(x)=2x+\int_0^{x^3}\sqrt{t^2+1}\,dt+\int_0^1\sqrt{s^2+1}ds$ is an increasing function on R.	26 June 7, 2003
6	Find the equation of the tangent to the curve $y=F(x)$ at $x=2$ where $F(x)=\int_4^{x^2}\frac{t+3}{t^2+4}dt$	
7	Let $f(x)=\int_0^{x^2}\frac{t-4}{t^2+1}\ dt$ Find the intervals on which f is increasing and the intervals on which f is decreasing	35 August 15, 2009

Homework

8	Let $f(x) = \int_{\cos x}^{\sin x} t^3\, dt$ 8 August 28, 1993 15 February 12.1996 Find an equation for the tangent line graph of f at $x = \frac{\pi}{4}$
9	Find the absolute maximum and the absolute minimum of function $f(x) = \int_2^x \left[\frac{16}{\sqrt{t}} - t^{\frac{2}{3}}\right] dt$ in the interval $[2, 25]$
10	Given that $2x^2 - 8 = \int_a^x f(t)\, dt$ 13 February 19, 1995 Find a formula for $f(x)$ and evaluate a
11	Find a continuous function f and a real number a 6 January 6, 1993 such that $\int_a^x 2f(t)dt = -1 + 2\sin x$ for $-\infty < x < \infty$
12	Let $f(x) = \int_1^x \sqrt{1+t^2}\, dt$ 17 January 8 ,1997 Show that $f(x) = 0$ has exactly one solution in $(-\infty, \infty)$
13	Let $f(x) = \int_x^{x+3} t(5-t)dt \;, x \in R.$ 32 August 02, 2008 Show that $f(x)$ has a maximum value at $x = 1$.
14	Find $\frac{dy}{dx}$ where $y = \int_1^{\cos x} \frac{1-u^2}{1+u^6}\, du$ 38 January 15, 2011

Homework

15	[4 pts.] Find the smallest positive critical number of $$f(x) = \int_0^{x^2} \cos\left(t^{3/2}\right) dt$$	39 5 June, 2011
16	(4 Points) Show that the function $$f(x) = \int_0^{1+x^3} (1 + \cos^2(\sqrt{t}))dt + \int_0^{8\pi} (t^2 \sin^5(t) + 1)dt$$ is increasing on $\mathbb{R}$	40 August 7, 2011
17	[3 Pts.] Evaluate the following integral $$\int_{-1}^{1} \left(t^3 \cos t + 2\sqrt{1-t^2}\right) dx$$	41 7 January 2012